Altering the symmetry of rings and square cylinders: First-occurring mode switching and non-linear evolution behaviour

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Abstract

The non-linear evolution of the subharmonic "mode C" instability behind rings of various aspect ratios is modelled by numerically evaluating coefficients of the Landau equation. The aspect ratio measures the ratio of the ring centreline diameter to the cross-section diameter. It is found that at small aspect ratios the mode C instability evolves through a supercritical (non-hysteretic) bifurcation, whereas at higher aspect ratios this behaviour changes to subcritical (hysteretic). This behaviour is rare: the well-known mode A and B instabilities, found in the wakes of various cylinders and rings, have been found to be consistently subcritical and supercritical, respectively. Thus the ring presents a convenient geometry to study the relationship between a wake and the non-linear characteristics of available instability modes.

Introduction

The study of three-dimensional transitions in bluff-body wakes received significant impetus following the seminal experiments of flow past a circular cylinder by Williamson (1988*a*, *b*). In these studies, the wake transitioned through two distinct regimes of three-dimensional flow as the Reynolds number (based on the freestream velocity and cylinder diameter, d) was increased. Firstly, as the Reynolds number increases beyond $Re \approx 180$, the formerly two-dimensional and time-periodic von Kármán vortex street behind the cylinder developed a spanwise-periodic three-dimensional structure now known as Mode A. This mode had a spanwise wavelength of approximately $4d$. A second mode, with a shorter spanwise wavelength of approximately $1d$, was detected with increasing prevalence over $230 < Re < 260$, before dominating the wake at higher Reynolds numbers. This second mode is now known as Mode B.

Two key numerical contributions of our understanding of these wakes emerged from the linear stability analysis of Barkley & Henderson (1996) and the direct numerical simulation of Modes A and B with a three-dimensional spectral-element algorithm by Thompson, Hourigan & Sheridan (1996). The analysis of Barkley & Henderson offered strikingly accurate predictions of the spanwise wavelength, spatio-temporal symmetry and critical Reynolds numbers for the onset of Modes A and B, and the simulations of Thompson *et al.* elucidated the detailed three-dimensional structure of the wakes arising from the predicted global instability modes.

Linear stability analysis is limited in that it only offers predictions valid for small-amplitude perturbations, and thus only reliably predicts the embryonic behaviour of a three-dimensional instability mode, rather than its later evolution and eventual saturation to some finite amplitude, which typically substantially alters the original flow. The Landau equation has been used with success to model the non-linear phase of evolving modes (e.g. see Henderson & Barkley 1996 and Henderson 1997 for the circular cylinder; Ghidersa & Dušek

2000 and Thompson, Leweke & Provansal 2001 for the sphere; Sheard, Thompson & Hourigan 2004 for rings). Conventionally, coefficients of the Landau equation are determined by manipulating a recorded time history of the amplitude of an evolving mode ($|A|$). Constructing a plot of $d(\log|A|)/dt$ against $|A|^2$ yields important mode characteristics: the *y*-axis intercept provides the growth rate of the mode in the linear regime, and the gradient at the *y*-axis intercept determines whether the mode evolves through a super- or sub-critical bifurcation. Analysis of the Landau equation demonstrates that a positive gradient permits bi-stability in the vicinity of the transition. This implies that the wake may exhibit hysteresis at the onset of the transition, consistent with a sub-critical bifurcation. Conversely, a negative gradient implies a supercritical bifurcation with no hysteresis at transition. Examples of modes which have been shown to exhibit subcritical (hysteretic) non-linear evolution behaviour include Mode A behind a circular cylinder and rings. Modes which exhibit supercritical behaviour include Mode B behind a circular cylinder and rings, as well as the first and second bifurcations behind a sphere.

While the Landau equation may be used to characterise the non-linear evolution of a mode, it is purely an amplitude evolution equation, and thus cannot provide insight into how the behaviour is linked to the fluid dynamics of the system in question. Furthermore, direct comparison between the wakes leading to super- and sub-critical bifurcations for the aforementioned modes is difficult as the geometries, and hence the underlying wake flows, are different. However, investigations into the wake behind a ring (Sheard, Thompson & Hourigan 2003; 2004) identified a third instability mode, known as Mode C. It will be shown in this study that Mode C displays either supercritical or subcritical properties, depending on the ring aspect ratio. The ring aspect ratio is defined as a ratio of the diameter of a circle denoting the centre of the ring cross-section, to the diameter of the circular cross-section itself. Thus the aspect ratio is a measure of the slenderness of the ring, with small aspect ratios describing thicker rings, and large aspect ratios describing slender rings. Local to the cross-section, the ring approaches a straight circular cylinder as the aspect ratio approaches infinity.

Hence the Mode C instability in the wake behind a ring allows the non-linear evolution of a three-dimensional instability mode to be investigated in a controlled and smoothly varying fashion, with the shift from supercritical to subcritical behaviour occurring as a function of aspect ratio. In this study, this mode will be investigated to determine the relationship between non-linear evolution characteristics and the properties of the underlying wake dynamics.

Methodology

Computations are carried out using an established and validated code (Sheard *et al.* 2007), which employs a nodal spectral-element discretization of the flow in cylindrical coordinates. Time integration is based on third-order backwards-differencing (Karniadakis, Israeli & Orszag 1991). Snapshots of time-periodic axisymmetric base flows are stored for later reconstruction using Fourier interpolation during stability analysis calculations. Linear stability analysis is conducted using a Krylov-based method involving the solution of leading modes of an eigensystem using an implicitly restarted Arnoldi method (Sheard, Fitzgerald & Ryan 2009). The linear stability analysis returns eigenvector fields corresponding to perturbation flow fields of global instability modes. These are used as initial conditions for

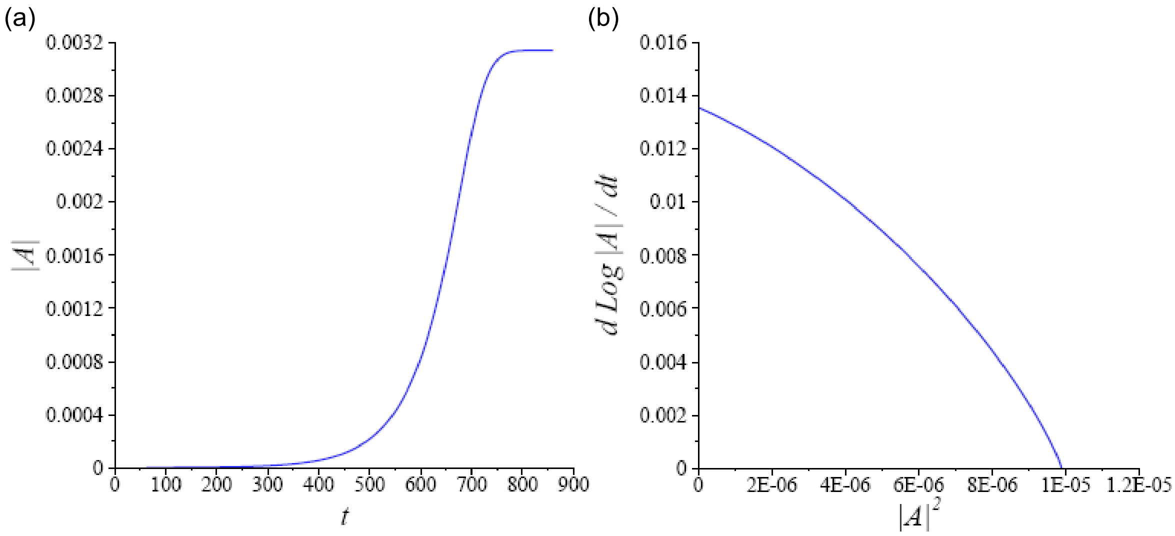

FIGURE 1: (a) The time history of an integrated amplitude norm of the Mode C instability behind a ring with AR = 10 at *Re* = 170, with azimuthal wavelength $\lambda_d = 1.7$. (b) A plot of d(log|A|)/d*t* against $|A|^2$ where the negative slope at the *y*-axis intercept indicates that the transition is supercritical.

three-dimensional direct numerical simulation, which is carried out using a spectral element-Fourier method following the formulation of Blackburn & Sherwin (2004).

To investigate the deviation from the linear (small-amplitude) regime occurring in the three-dimensional wakes, appropriately amplified versions of the perturbation fields used as initial conditions for the three-dimensional simulations are subtracted from the solutions evolved over several periods. The remaining structures in the solution correspond to the initial non-linear disturbances in the flow. These are then analysed across different aspect ratios.

Results

At an aspect ratio AR = 5, the evolution of a three-dimensional wake with an azimuthal wavelength of approximately 1.7*d* is computed using the predicted Mode C instability as an initial condition. The resulting non-linear evolution to saturation is shown in Figure 1, where the negative gradient of at the *y*-axis of the plot in Figure 1(b) demonstrates that the bifurcation is supercritical.

At an aspect ratio AR = 10, the evolution is computed and the behaviour is displayed in Figure 2. The positive gradient at the *y*-axis of the plot in Figure 2(b) demonstrates that the bifurcation is subcritical. Hence the Mode C instability undergoes a transition from supercritical behaviour at small aspect ratios (including AR = 5) to subcritical behaviour at larger aspect ratios (including AR = 10).

The presentation supporting this paper will explore the following questions: At what aspect ratio does the bifurcation arising from the Mode C instability shift from supercritical to subcritical? How does the region of hysteresis at higher aspect ratios change with ring aspect ratio? What is the relationship between the locations and structure of non- linear velocity disturbances in the early stages of mode evolution, and how do these relate to both aspect ratio and the non-linear evolution characteristics?

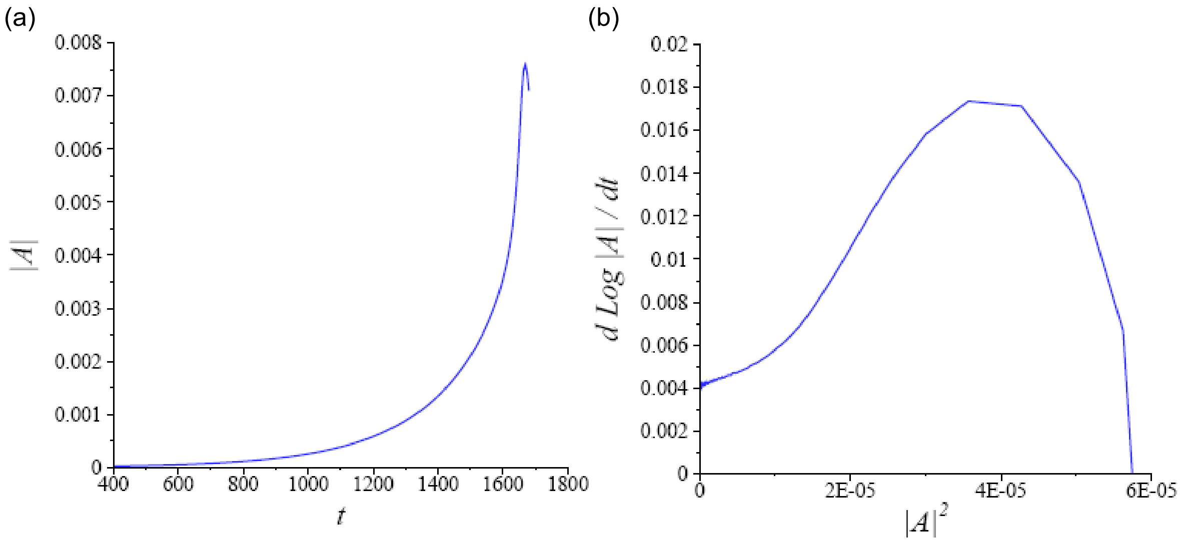

FIGURE 2: (a) The time history of an integrated amplitude norm of the Mode C instability behind a ring with AR = 10 at *Re* = 230, with azimuthal wavelength $\lambda_d = 1.7$. (b) A plot of d(log|A|)/d*t* against $|A|^2$ where the negative slope at the *y*-axis intercept indicates that the transition is supercritical.

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